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OSCILLATORY BEHAVIOR OF FIRST ORDER DELAY DIFFERENCE EQUATION

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ABSTRACT

We established some sufficient conditions involving the oscillatory behavior of first order delay difference equation of the form

$\Delta(a_n w_n) + r_n w_{n-m} = 0, n \in N.$

Examples are provided to illustrate the results.

Keywords: Difference Equation, Oscillation, Delay.

INTRODUCTION

Consider the first order delay difference equation,

$$\Delta(a_n w_n) + r_n w_{n-m} = 0, n \in N. \tag{1.1}$$

where $\{a_n\}, \{r_n\}$ are sequences of positive real numbers, m is a non-negative integers.A nontrivial solution $\{w_n\}$ of (1.1) is oscillatory if it is neither eventually positive nor eventually negative and non-oscillatory otherwise.

MAIN RESULTS

Theorem 2.1. Let $r_n > 0$ for all $n \in N$ and

$\limsup_{n\to\infty} \sum_{s=n-m}^{n} \frac{r_s}{a_s} > 1$(2.1)

Then, the difference equation (1.1) is oscillatory.

Proof. Let $w_n > 0$ for all $n \in N(n_1)$ be a solution of (1.1).

Since $r_n > 0$, for all $n \in N(n_1 + m)$, equation (1.1) implies that $\Delta w_n \leq 0$, and hence w_n is non-increasing on $N(n_1 + m)$.

Therefore, $\lim_{n\to\infty} w_n = \gamma \geq 0$ exists.

But taking the limit in (1.1) ensures that $\gamma = 0$.

Now summing (1.1) from $n_2 \in N(n_1 + m)$, to $n_2 + m$, we have

$$a_{(n_2+m+1)} w_{(n_2+m+1)} - a_{n_2} w_{n_2} + \sum_{s=n_2}^{n_2+m} \frac{r_s}{a_s} w_{s-m} = 0,$$

which implies that

$$a_{(n_2+m+1)} w_{(n_2+m+1)} - a_{n_2} w_{n_2} \left[1 - \sum_{s=n_2}^{n_2+m} \frac{r_s}{a_s}\right] \leq 0.$$

Therefore,

$$1 - \sum_{s=n_2}^{n_2+m} \frac{r_s}{a_s} \geq 0.$$

and hence

$$1 \geq \limsup_{n_2\to\infty} \sum_{s=n_2}^{n_2+m} \frac{r_s}{a_s}.$$

This contradicts (2.1) and completes the proof.

Theorem 2.2. Suppose that

$$\liminf_{n\to\infty} \frac{r_n}{a_n} = d > 0 \; and \; \limsup_{n\to\infty} \frac{r_n}{a_n} > 1 - d. \tag{2.2}$$

Then, the following hold

(i) $a_{n+1}x_{n+1} - a_n x_n + \frac{r_n}{a_n} x_{n-m} \leq 0, n \in N$ (2.3)

has no eventually positive solution

$(ii) a_{n+1}z_{n+1} - a_n z_n + \frac{r_n}{a_n} z_{n-m} \geq 0, n \in N$ (2.4)

has no eventually negative solution

(iii)difference equation (1.1) is oscillatory.

Proof. Assume that x_n is an eventually positive solution of (2.3), that is there exists a $n_1 \epsilon N(1)$ such that $x_n > 0$ for all $n \in N(n_1)$.

Let $\epsilon > 0, 0 < \epsilon < d$ and $n_2 \geq n_1$ be such that

$\frac{r_n}{a_n} \geq d - \epsilon > 0,$ for all $n \in N(n_2)$.

Let $n_3 = \max\{n_1 + m, n_2\}$

so that

$$a_n x_n \geq \frac{r_n}{a_n} x_{n-m} \geq (d-\epsilon) x_{n-1}, \text{ for all } n \in N(n_3),$$

since x_n is non-increasing for all $n \epsilon N(n_3)$.

On the other hand, we have

$$0 \geq a_{n+1}x_{n+1} - a_n x_n + \frac{r_n}{a_n} x_{n-m}$$

$$\geq a_{n+1}x_{n+1} + a_n x_n \left(\frac{r_n}{a_n} - 1\right), \text{ for all } n \in N(n_3).$$

so that

$$a_n x_n \left(\frac{r_n}{a_n} - 1 + d - \epsilon\right) \leq 0, \text{ for all } n \in N(n_3),$$

Thus, it follows that

$$\frac{r_n}{a_n} \leq 1 - d + \epsilon, \text{ for all } n \in N(n_3),$$

and hence

$$\lim_{n\to\infty} \sup \frac{r_n}{a_n} \leq 1 - d + \epsilon$$

However, since $\epsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} \sup \frac{r_n}{a_n} \leq 1 - d.$$

This contradicts (2.2) and the proof of (i) is complete.

The conclusion (ii) follows from (i) by letting $w_n = -z_n$ for an eventually negative solution x_n of (2.4).

Finally, (iii) follows from (i) and (ii).

Theorem 2.3. Assume that $r_n > 0$ for all $n \in N$ and

$$\lim_{n\to\infty} \inf \sum_{s=n-m}^{n-1} \frac{r_s}{a_s} > \frac{m^{m+1}}{(m+1)^{m+1}} . \quad (2.5)$$

Then, the conclusions of Theorem (2.2) hold.

Proof. We shall prove only (iii), whereas (i) and (ii) can be proved analogously.

Let w_n be a non-oscillatory solution of (1.1), which we can assume to be positive eventually, and since $r_n > 0$ this solution $w_n > 0$ is eventually decreasing.

Therefore, on using $w_n \leq w_{n-m}$ in (1.1), eventually we obtain

$$\frac{r_n}{a_n} \leq 1 - \frac{w_{n+1}}{w_n}.$$

and hence on using arithmetic and geometric means inequality, we find

$$\frac{1}{m}\sum_{s=n-m}^{n-1}\frac{r_s}{a_s} \leq 1-\frac{1}{m}\sum_{s=n-m}^{n-1}\frac{w_{s+1}}{w_s}$$
$$\leq 1-\left[\frac{w_n}{w_{n-m}}\right]^{\frac{1}{m}}. \quad (2.6)$$

setting $\gamma = \frac{m^m}{(m+1)^{m+1}}$,
from(1.1) we can choose a constant δ such that for n sufficiently large

$$\gamma < \delta \leq \left(\frac{1}{m}\right)\sum_{s=n-m}^{n-1}\frac{r_s}{a_s}.$$

Therefore, from (2.6) for all large n,

$$\left[\frac{w_n}{w_{n-m}}\right]^{\frac{1}{m}} \leq 1-\delta,$$

which in particular implies that $0 < \delta < 1$.
Now since

$$\max_{0\leq\eta\leq 1}\left[(1-\eta)\eta^{\frac{1}{m}}\right] = \gamma^{\frac{1}{m}}.$$

we have $1-\eta \leq \gamma^{\frac{1}{m}}\eta^{\frac{-1}{m}}$for $0 < \eta \leq 1$,
and hence it follows that

$$\left[\frac{w_n}{w_{n-m}}\right]^{\frac{1}{m}} \leq \gamma^{\frac{1}{m}}\delta^{\frac{-1}{m}}$$

which is the same as

$$\frac{\delta}{\gamma}w_n \leq w_{n-m}. \quad (2.7)$$

Now using (2.7) instead of $w_n \leq w_{n-m}$ in (1.1) and repeating the arguments, we find

$$\left(\frac{\delta}{\gamma}\right)^2 w_n \leq w_{n-m} \text{ for all large } n. \quad (2.8)$$

Thus, by induction, for every $n \in N(1)$ there exists an integer n_n such that for all $n \in N(n_n)$

$$\left(\frac{\delta}{\gamma}\right)^n w_n \leq w_{n-m}. \quad (2.9)$$

Next, for sufficiently large n,

$$\sum_{s=n-m}^{n}\frac{r_s}{a_s} \geq \sum_{s=n-m}^{n-1}\frac{r_s}{a_s} \geq m\delta = M. \quad (2.10)$$

say, since $\delta > \gamma$,
we can choose n such that

$$\left(\frac{\delta}{\gamma}\right)^n > \left(\frac{2}{M}\right)^2 \quad . \quad (2.11)$$

For this specific value of n, we consider n sufficiently large, say n^* so that for all $n \geq n^*$, all the above inequalities are satisfied.
Then, for each $n \geq n^* + m$ there exists an integer $\hat{n}$ with $n-m \leq \hat{n} \leq n$ so that

$$\sum_{s=n-m}^{\hat{n}}\frac{r_s}{a_s} \geq \left(\frac{M}{2}\right) \text{ and } \sum_{s=\hat{n}}^{n}\frac{r_s}{a_s} \geq \left(\frac{M}{2}\right).$$

From (1.1) and the non-increasing nature of w_n, we have

$$-w_{n-m} \leq w_{\hat{n}+1} - w_{n-m}$$
$$= \sum_{s=n-m}^{\hat{n}} w_{s+1} - w_s$$
$$= -\sum_{s=n-m}^{\hat{n}}\frac{r_s}{a_s}w_{s-m}$$

$$\leq -\left[\sum_{s=n-m}^{\hat{n}} \frac{r_s}{a_s}\right] w_{\hat{n}-m}$$
$$\leq -\frac{M}{2} w_{\hat{n}-m}$$

and hence

$$\frac{M}{2}(w_{\hat{n}-m}) \leq w_{n-m.} \tag{2.12}$$

Similarly, we find

$$-w_{\hat{n}} \leq w_{n+1} - w_{\hat{n}}$$
$$= \sum_{s=\hat{n}}^{n} \left((w_{s+1}) - w_s\right)$$
$$= -\sum_{s=\hat{n}}^{n} \frac{r_s}{a_s} w_{n-m}$$
$$\leq -\left[\sum_{s=\hat{n}}^{n} \frac{r_s}{a_s}\right] w_{s-m}$$
$$\leq -\frac{M}{2} w_{n-m}$$

and so

$$\frac{M}{2} w_{n-m} \leq w_{\hat{n}} \ . \tag{2.13}$$

Combining (2.9),(2.12)and (2.13), we get

$$\left(\frac{\delta}{\gamma}\right)^n \leq \frac{w_{\hat{n}-m}}{w_{\hat{n}}} \leq \left(\frac{2}{M}\right)^2 . \tag{2.14}$$

But this contradicts (2.11) and the proof is complete.

3. EXAMPLE

Example 3.1.Consider the first order delay difference equation

$$\Delta(nw_n) + (2n+1)w_{n-2} = 0 \tag{3.1}$$

Here $a_n = n, r_n = 2n + 1$.

All conditions of Theorem(2.3) are satisfied.

Hence all solution of equation (3.1) are oscillatory.

In fact, $w_n = (-1)^n$ is one such solution of equation(3.1).

Example 3.2. Consider the first order delay difference equation

$$\Delta\left(\frac{1}{n}\right) + (2n+1)w_{n-2} = 0 \tag{3.2}$$

Here $a_n = \frac{1}{n}, r_n = 2n + 1$.

All conditions of Theorem(2.3) are satisfied.

Hence all solution of equation (3.2) are oscillatory.

In fact, $w_n = (-1)^n$ is one such solution of equation (3.2).

4. Conclusion

In this paper, the oscillatory behaviour of first order delay difference equation of the form(1.1) has been studied and established some sufficient condition for oscillatory behavior of (1.1).Various example are considered to illustrate the main results.

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